

Homework 1

3.2) Hamiltonian: $H = \int \frac{d^3k}{(2\pi)^3 \omega_k} a^\dagger(\vec{k}) a(\vec{k}) \omega_k$

\Rightarrow Clearly, $H|0\rangle = 0$ since $a(\vec{k})|0\rangle = 0$

\Rightarrow Consider $H a^\dagger(\vec{p}_1)|0\rangle = \int \frac{d^3k \omega_k}{(2\pi)^3 \omega_k} a^\dagger(\vec{k}) a(\vec{k}) a^\dagger(\vec{p}_1)|0\rangle$

as an example

$\underbrace{a^\dagger(\vec{k}) a(\vec{k})}_{\text{commute these}} a^\dagger(\vec{p}_1)|0\rangle = 0$, get

$$\int \frac{d^3k \omega_k a^\dagger(\vec{k})}{(2\pi)^3 \omega_k} 2\omega_{p_1} \delta^{(3)}(\vec{p}_1 - \vec{k}) (2\pi)^3 |0\rangle$$

$$= \omega_1 a^\dagger(\vec{p}_1)|0\rangle \Rightarrow \text{eigenstate, with eigenvalue}$$

$$\omega_1 = \sqrt{\vec{p}_1^2 + m^2}$$

\Rightarrow Consider $H a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2)|0\rangle =$

$$\int \frac{d^3k}{(2\pi)^3} a^\dagger(\vec{k}) a(\vec{k}) a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2)|0\rangle$$

$$= \omega_1 a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2)|0\rangle + \int \frac{d^3k}{(2\pi)^3} a^\dagger(\vec{k}) a^\dagger(\vec{p}_1) a(\vec{k}) a^\dagger(\vec{p}_2)|0\rangle$$

$\underbrace{a^\dagger(\vec{k}) a(\vec{k})}_{\text{replace with}} (2\pi)^3 \omega_2 \delta^{(3)}(\vec{k} - \vec{p}_2)$

$$= (\omega_1 + \omega_2) a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2)|0\rangle$$

\Rightarrow

Consider $H a_1^\dagger \dots a_n^\dagger|0\rangle$ now

$$= \int \frac{d^3k}{(2\pi)^3} a^+(\vec{k}) \underbrace{a(\vec{k})}_{\text{systematically move } a(\vec{k}) \text{ to the right. Get two terms each time}} a^+_1 \dots a^+_n |0\rangle$$

$$w_i a^+_1 a^+_2 \dots a^+_{i-1} a^+_{i+1} \dots a^+_n |0\rangle$$

$$+ \int \frac{d^3k}{(2\pi)^3} a^+(\vec{k}) a^+_1 \dots a^+_i a(\vec{k}) a^+_{i+1} \dots a^+_n |0\rangle$$

For $i=1$, 1st term is $w_i a^+_1 a^+_2 \dots a^+_n |0\rangle$ - second term is $\int \frac{d^3k}{(2\pi)^3} a^+(\vec{k}) a^+_1 a(\vec{k}) a^+_2 \dots a^+_n |0\rangle$

keep doing for $i=1$ to $n-1$, generate

$$\left[\sum_{i=1}^{n-1} w_i \right] a^+_1 \dots a^+_n |0\rangle + \int \frac{d^3k}{(2\pi)^3} a^+(\vec{k}) a^+_1 \dots a^+_{n-1} a(\vec{k}) \underbrace{a^+_n}_{\text{commute once more, } a(\vec{k})|0\rangle = 0} |0\rangle$$

$$= (w_1 + \dots + w_n) a^+_1 \dots a^+_n |0\rangle \Rightarrow \text{as desired}$$

$$3.5) \quad \mathcal{L} = \partial_\mu \phi \partial^\mu \phi^+ - m^2 \phi \phi^+ + \mathcal{R}_0$$

(a) Lagrange's eq: $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right\}$

$$\Rightarrow -m^2 \phi^+ = \partial^2 \phi^+ \Rightarrow \{\partial^2 + m^2\} \phi^+ = 0$$

Take complex conjugate $\Rightarrow \{\partial^2 + m^2\} \bar{\phi} = 0$

(b) $\Pi = \frac{\partial \mathcal{L}}{\partial (\dot{\phi})} = \dot{\phi}^+$ while $\Pi^+ = \frac{\partial \mathcal{L}}{\partial (\dot{\phi}^+)} = \dot{\phi}$
 \Rightarrow as required $(\Pi)^+ = \Pi^+$

$$\mathcal{H} = \vec{d}^3 \times \mathcal{H} \Rightarrow \mathcal{H} = \Pi \dot{\phi} + \Pi^+ \dot{\phi}^+ - \mathcal{L}$$

$$\mathcal{H} = \Pi \Pi^+ + \nabla \phi \cdot \nabla \phi^+ + m^2 \phi \phi^+ - \mathcal{R}_0$$

(c) Begin with mode expansion

$$\phi(x) = \int \frac{d^3 k}{(2\pi)^3 2\omega} \left\{ a(\vec{k}) e^{-ik \cdot x} + b^+(\vec{k}) e^{ik \cdot x} \right\}$$

$$\phi^+(x) = \int \frac{d^3 k}{(2\pi)^3 2\omega} \left\{ b(\vec{k}) e^{-ik \cdot x} + a^+(\vec{k}) e^{ik \cdot x} \right\}$$

$$\dot{\phi}(x) = \int \frac{d^3 k}{(2\pi)^3 2} \left\{ -i a(\vec{k}) e^{-ik \cdot x} + i b^+(\vec{k}) e^{ik \cdot x} \right\}$$

$$\dot{\phi}^+(x) = \int \frac{d^3 k}{(2\pi)^3 2} \left\{ -i b(\vec{k}) e^{-ik \cdot x} + i a^+(\vec{k}) e^{ik \cdot x} \right\}$$

$$\begin{aligned}
 \text{Note: } & \int d^3x \left\{ \left(\frac{\partial}{\partial t} e^{ik \cdot x} \right) \phi(x) - e^{ik \cdot x} \frac{\partial}{\partial t} \phi(x) \right\} \\
 &= \frac{1}{2} \int d^3x \frac{d^3k'}{(2\pi)^3} \left\{ i[a(\vec{k}') e^{i(k-k') \cdot x} + b^+(\vec{k}') e^{i(k+k') \cdot x}] \right. \\
 &\quad \left. - [-ia(\vec{k}') e^{i(k-k') \cdot x} + ib^+(\vec{k}') e^{i(k+k') \cdot x}] \right\} \\
 &= i a(\vec{k}) \\
 \Rightarrow a(\vec{k}) &= -i \int d^3x \left\{ \left(\frac{\partial}{\partial t} e^{ik \cdot x} \right) \phi(x) - e^{ik \cdot x} \dot{\phi}(x) \right\} \\
 &= i \int d^3x e^{ik \cdot x} \left\{ \dot{\phi}(x) - i\omega \phi(x) \right\}
 \end{aligned}$$

Since ϕ^+ can be obtained from ϕ via interchanging a, b ,

$$b(\vec{k}) = i \int d^3x e^{ik \cdot x} \left\{ \dot{\phi}^+(x) - i\omega \phi^+(x) \right\}$$

$$\begin{aligned}
 @) \text{ impose } & [\phi(\vec{x}, t), \Pi(\vec{x}', t)] = i \delta^{(3)}(\vec{x} - \vec{x}') \\
 &= [\phi(\vec{x}, t), \dot{\phi}^+(\vec{x}', t)] \\
 &= [\phi^+(\vec{x}, t), \Pi^+(\vec{x}', t)] = i \delta^{(3)}(\vec{x} - \vec{x}') \quad \text{also } [\phi(\vec{x}, t), \Pi^+(\vec{x}', t)] = 0 \\
 &= [\phi^+(\vec{x}, t), \dot{\phi}(\vec{x}', t)] \quad \text{if } [\phi^+(\vec{x}, t), \Pi(\vec{x}', t)] = 0 \\
 &[\phi(\vec{x}, t), \phi^+(\vec{x}', t)] = [\Pi(\vec{x}, t), \Pi^+(\vec{x}', t)] = 0
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow [a(\vec{k}), a(\vec{k}')] &= - \int d^3x d^3y e^{ik \cdot x} e^{ik' \cdot y} \\
 &\quad [\Pi^+(x) - i\omega \phi(x), \Pi^+(y) - i\omega' \phi(y)] \\
 &= 0
 \end{aligned}$$

$$\text{Similarly, } [b(\vec{k}), b(\vec{k}')] = 0$$

$$\Rightarrow [\alpha(\vec{k}), \alpha^+(\vec{k}')] = + \int d^3x d^3y e^{ik_x x - ik'_x y} [\Pi^+(x) - i\omega \phi(x), \Pi(y) + i\omega' \phi^+(y)] \\ = + \int d^3x d^3y e^{ik_x x - ik'_x y} \left\{ \begin{array}{l} -i\omega [\phi(x), \Pi(y)] \\ -i\omega [\phi^+(y), \Pi^+(x)] \end{array} \right\} \\ \Rightarrow x^0 = y^0, \text{ so this becomes } 2\omega (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$$

$$\Rightarrow [\beta(\vec{k}), \beta^+(\vec{k}')] = \int d^3x d^3y e^{ik_x x - ik'_x y} [\Pi(x) - i\omega \phi^+(x), \Pi^+(y) + i\omega' \phi(y)] \\ = 2\omega (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$$

$$\Rightarrow [\alpha(\vec{k}), \beta(\vec{k}')] = - \int d^3x d^3y e^{ik_x x + ik'_x y} [\Pi^+(x) - i\omega \phi(x), \Pi(y) - i\omega' \phi^+(y)] \\ = - \int d^3x d^3y e^{ik_x x + ik'_x y} \left\{ \begin{array}{l} -i\omega [\phi(x), \Pi(y)] \\ + i\omega' [\phi^+(y), \Pi^+(x)] \end{array} \right\} \\ = - \int d^3x e^{ik_x x + ik'_x x} \left\{ \begin{array}{l} -i\omega + i\omega' \end{array} \right\} = 0$$

$$\Rightarrow \text{Similarly, } [\alpha^+(\vec{k}), \beta^+(\vec{k}')] = 0$$

$$\Rightarrow [\alpha(\vec{k}), \beta^+(\vec{k}')] = \int d^3x d^3y e^{ik_x x - ik'_x y} [\Pi^+(x) - i\omega \phi(x), \Pi^+(y) + i\omega' \phi^+(y)] = 0$$

$$\text{Similarly, } [\alpha^+(\vec{k}), \beta(\vec{k}')] = 0$$

Four terms in H

$$(1) \pi\pi^+ = \int d^3x \left(\frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \right) \frac{1}{4} \left\{ -b(\vec{k})a(\vec{k}')e^{-i(k+k')\cdot x} \right. \\ -a^+(\vec{k})b^+(\vec{k}')e^{i(k+k')\cdot x} +a^+(\vec{k})a(\vec{k}')e^{i(k-k')\cdot x} \\ \left. +b(\vec{k})b^+(\vec{k})e^{-i(k-k')\cdot x} \right\} \\ = \frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \left\{ a^+(\vec{k})a(\vec{k}) + b(\vec{k}')b^+(\vec{k}') - b(\vec{k})a(-\vec{k})e^{-2i\omega t} \right. \\ \left. -a^+(\vec{k})b^+(-\vec{k})e^{2i\omega t} \right\}$$

$$(2) \nabla\phi \cdot \nabla\phi^+ = \int d^3x \left(\frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \right) \frac{1}{4\omega\omega'} \left\{ -\vec{k}\cdot\vec{k}'a(\vec{k})b(\vec{k}')e^{-i(k+k')\cdot x} \right. \\ -\vec{k}\cdot\vec{k}'b^+(\vec{k})a^+(\vec{k}')e^{i(k+k')\cdot x} +\vec{k}\cdot\vec{k}'a(\vec{k})a^+(\vec{k}')e^{-i(k-k')} \\ \left. +\vec{k}\cdot\vec{k}'b^+(\vec{k})b(\vec{k}')e^{i(k-k')\cdot x} \right\} \\ = \frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \frac{\vec{k}^2}{\omega^2} \left\{ a(\vec{k})a^+(\vec{k}) + b^+(\vec{k})b(\vec{k}) + a(\vec{k})b(-\vec{k})e^{-2i\omega t} \right. \\ \left. + b^+(\vec{k})a^+(-\vec{k})e^{2i\omega t} \right\}$$

$$(3) m^2\phi\phi^+ : \frac{m^2}{4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega^2} \left\{ a(\vec{k})a^+(\vec{k}) + b^+(\vec{k})b(\vec{k}) + a(\vec{k})b(-\vec{k})e^{-2i\omega t} \right. \\ \left. + b^+(\vec{k})a^+(-\vec{k})e^{2i\omega t} \right\}$$

Sum these three:

$$\frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \left\{ a^+(\vec{k})a(\vec{k}) + b(\vec{k})b^+(\vec{k}) \right. \\ \left. + \frac{\vec{k}^2+m^2}{\omega^2} [a(\vec{k})a^+(\vec{k}) + b^+(\vec{k})b(\vec{k})] \right\} \\ + e^{2i\omega t} \left[-a^+(\vec{k})b^+(-\vec{k}) + \left(\frac{\vec{k}^2+m^2}{\omega^2} \right) b^+(\vec{k})a^+(-\vec{k}) \right] \\ + e^{-2i\omega t} \left[-b(\vec{k})a(-\vec{k}) + \left(\frac{\vec{k}^2+m^2}{\omega^2} \right) a(\vec{k})b(-\vec{k}) \right]$$

Use $\vec{u} \rightarrow -\vec{u}$ in last two lines

$$\Rightarrow H = \frac{1}{4} \int \frac{d^3 k}{(2\pi)^3} \left\{ a^+(-\vec{k}) a(\vec{k}) + b^+(-\vec{k}) b(\vec{k}) \right. \\ \left. + a(-\vec{k}) a^+(\vec{k}) + b(-\vec{k}) b^+(\vec{k}) \right\} \\ - \{ d^3 x \} \mathcal{D}_0$$

Normal-order:

$$H = \int \frac{d^3 k}{(2\pi)^3 2\omega} \omega \left\{ a^+(-\vec{k}) a(\vec{k}) + b^+(-\vec{k}) b(\vec{k}) \right\} \\ + \frac{1}{4} \int \frac{d^3 k}{(2\pi)^3} 2\omega (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') - \{ d^3 x \} \mathcal{D}_0$$

Set $\{ d^3 x \} = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') = \checkmark$; focus on

last two lines

$$\checkmark \left\{ \frac{1}{8\pi^3} \left\{ d^3 k \sqrt{\vec{k}^2 + m^2} - \mathcal{D}_0 \right\} \right\} = \checkmark \left\{ \frac{1^4}{8\pi^2} - \mathcal{D}_0 \right\} \\ \cong 4\pi \int_0^1 dk k^3 = \pi \lambda^4 \quad \Rightarrow \mathcal{D}_0 = \frac{\lambda^4}{8\pi^2} \text{ to cancel this}$$

4.v) Consider the integral

$$I(x) = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} e^{ik \cdot x} \quad \text{where } x^2 < 0 \quad (\text{Note } x^2 = (x^0)^2 - \vec{x}^2)$$

By Lorentz invariance of the exponential, can evaluate in a frame where $x = (0, \vec{x})$

$$\Rightarrow I(x) = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} e^{i\vec{k} \cdot \vec{x}} \quad \begin{aligned} &\text{Align } \vec{x} \text{ along } z\text{-axis of } k \text{ integral} \\ &\vec{k} \cdot \vec{x} = |\vec{k}| |\vec{x}| \cos \theta \end{aligned}$$

$$= \frac{1}{16\pi^3} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \int_0^\infty dk \frac{k^2}{\sqrt{k^2 + m^2}} e^{-ikr \cos \theta} \quad \begin{aligned} &\text{with} \\ &r = |\vec{x}| \end{aligned}$$

$$= \frac{1}{8\pi^2} \int_{-1}^1 dx \int_0^\infty dk \frac{k^2}{\sqrt{k^2 + m^2}} e^{-ikr x}$$

$$= \frac{-i}{8\pi(r)} \int_0^\infty dk \frac{k}{\sqrt{k^2 + m^2}} \left\{ e^{-ikr} - e^{ikr} \right\}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dk \frac{k \sin(kr)}{\sqrt{k^2 + m^2}}$$

To evaluate, note From your favorite integral table (Abramowitz - Stegun A.6.25) that

$$K_0(xz) = \int_0^\infty dt \frac{\cos(xt)}{\sqrt{t^2 + z^2}}$$

$$\text{Also, } \frac{d K_0(y)}{dy} = -K_1(y) \quad (\text{A+S A.6.27})$$

$$\Rightarrow K_1(xr) = \frac{1}{2} \int_0^\infty dt \frac{t \sin(xt)}{\sqrt{t^2 + r^2}} \quad \text{Apply with } \begin{matrix} x=r \\ z=m \end{matrix}$$

$$\Rightarrow J(x) = \frac{m}{4\pi^2 r} K_1(mr) \quad \text{By L.I, } r = \sqrt{|x|^2} \Rightarrow \sqrt{-t^2 + |x|^2} \\ = \sqrt{-x^2}$$

Note the series expansion $K_1(mr) = \frac{1}{mr} + O(mr)$
 $(A-S d.6.9)$

$$\Rightarrow I_{m \rightarrow 0}(x) = \frac{1}{4\pi^2 r^2}$$

(1) From our solution in HW #1, we can write the lowering operators for the two particle types appearing in the complex scalar field as

$$a(k) = i \int d^3x e^{ik \cdot x} \{ \dot{\phi}(x) - i\omega \phi(x) \}$$

$$b(k) = i \int d^3x e^{-ik \cdot x} \{ \dot{\phi}^+(x) - i\omega \phi^+(x) \}$$

By analogy with our derivation in class, we can derive the expressions

$$a_i^+(+\infty) - a_i^+(-\infty) = i \int d^3k F_i(\vec{k}) \int d^4x e^{ik \cdot x} \{ \partial^2 + m^2 \} \phi(x)$$

$$b_i^+(+\infty) - b_i^+(-\infty) = i \int d^3k F_i(\vec{k}) \int d^4x e^{-ik \cdot x} \{ \partial^2 + m^2 \} \phi^+(x)$$

$$\text{with } a_i^+(t) = \int d^3k F_i(\vec{k}) a^+(\vec{k})$$

$$b_i^+(t) = \int d^3k F_i(\vec{k}) b^+(\vec{k})$$

The derivation of the LSZ reduction formula is identical to that in class, except that $\phi^+(x)$ is substituted for the particle b^+

$$\Rightarrow \langle F | i \rangle = (-i)^{n_i} \int d^4x_1 \dots d^4x_{n_i} d^4x'_1 \dots d^4x'_{n'_F} (-i)^{n_i} \int \dots \int$$

$$e^{ik_1 \cdot x_1} \dots e^{ik_{n_i} \cdot x_{n_i}} e^{-ik'_1 \cdot x'_1} \dots e^{-ik'_{n'_F} \cdot x'_{n'_F}}$$

$$(\partial_1^2 + m^2) \dots (\partial_{n_i}^2 + m^2) (\partial'_1^2 + m^2) (\partial'_{n'_F}^2 + m^2)$$

$$\langle 0 | T \{ \phi_1(x_1) \dots \phi_{n_i}(x_{n_i}) \phi'_1(x'_1) \dots \phi'_{n'_F}(x'_{n'_F}) \} | 0 \rangle$$

The primes are associated with outgoing particles.
This is for no incoming & no final particles.

$\phi_i'(x_i)$ is $\phi(x_i)$ if particle i is created by a_i^+ ,
or $\phi^+(x_i)$ if it is created by b_i^+

